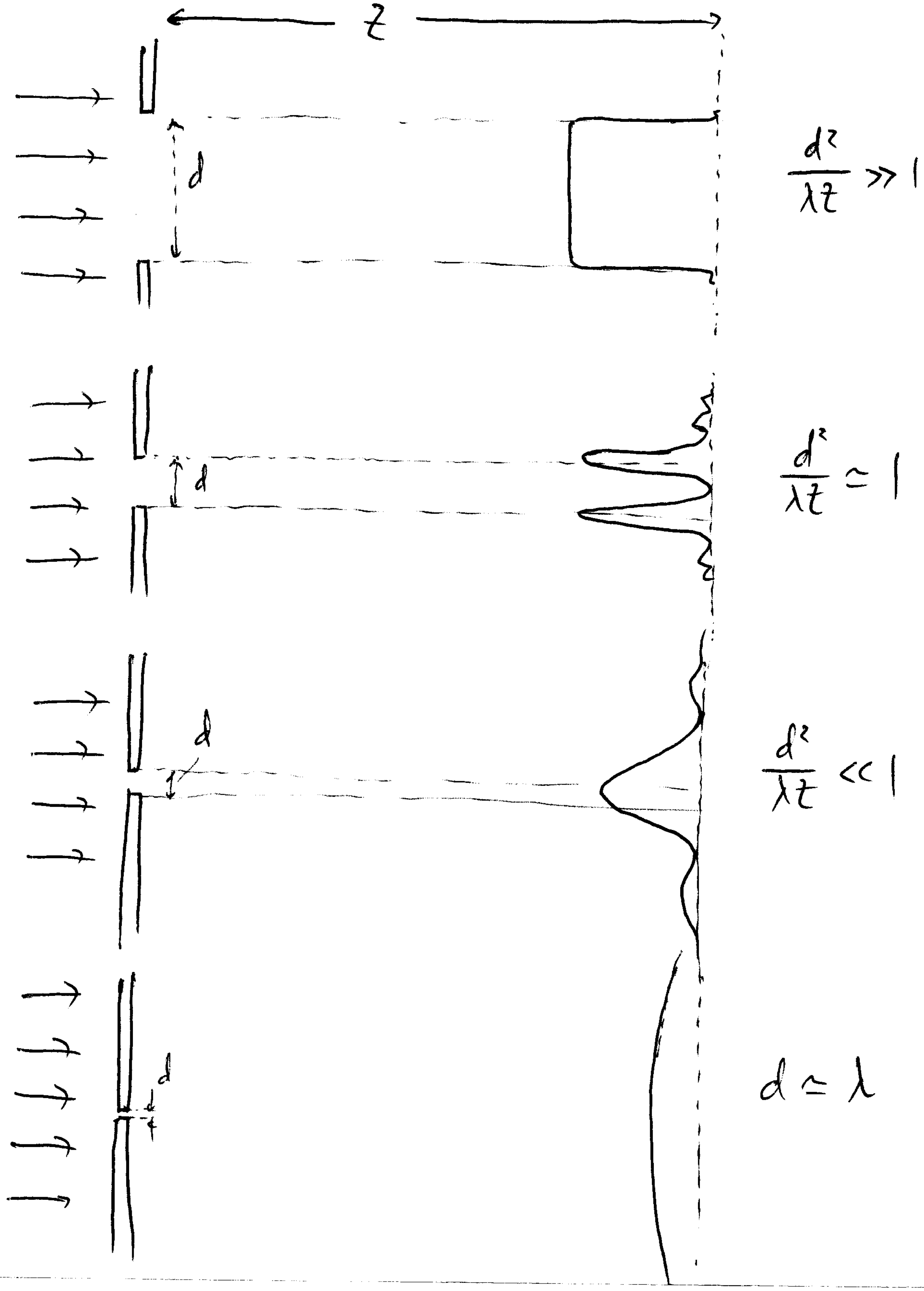
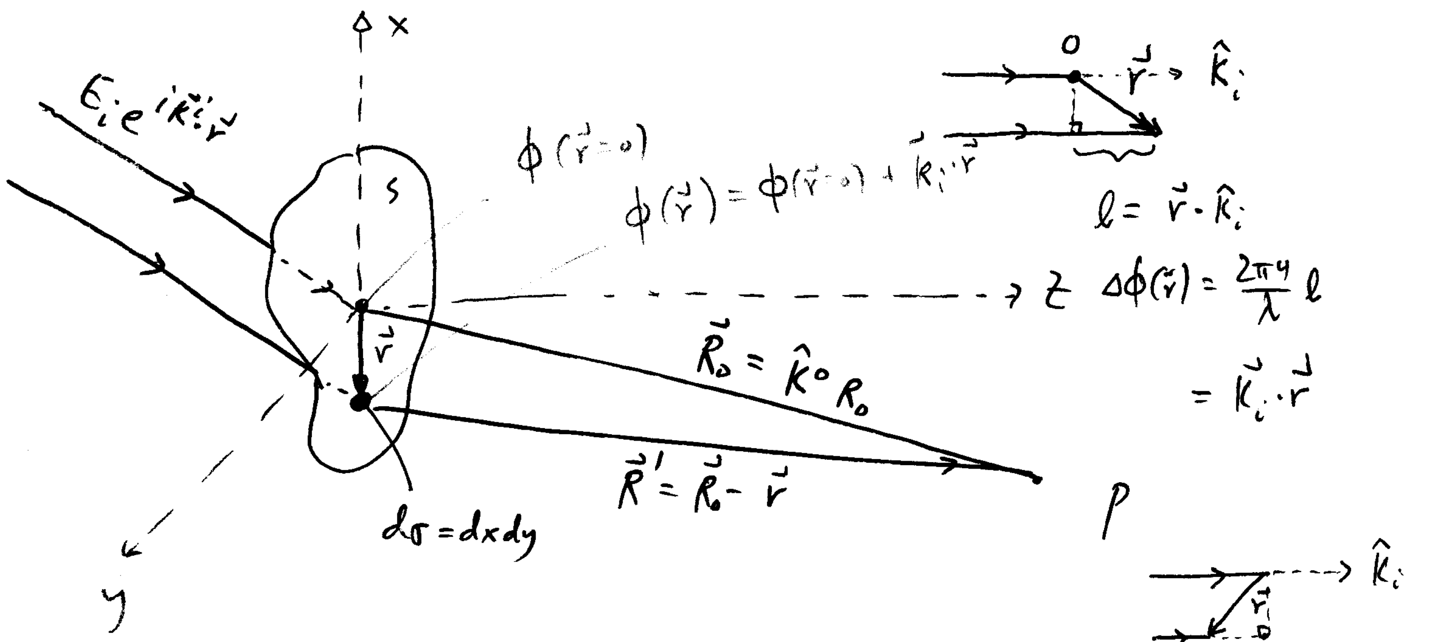
Crossover from geometric optics to diffraction optics



Mathematical result of Huygen's principle:



$$-l=\vec r\cdot\hat k_i$$

$$\Delta\phi(\vec r)=\frac{2\pi}{\lambda_0}(-l)=\vec k_i\cdot\vec r$$

$$E_0(P)=\frac{k^i e^{i\phi(\vec r=0)}}{2\pi i}\int_S d\sigma\,\frac{e^{ik^0 R'}}{|\vec R'|}\,e^{i\vec k^i\cdot\vec r}\cdot E_i$$

Normalization factor — area element — Huygen's wavelets — initial phase of Huygen's wavelets — initial amplitude of Huygen's wavelets.

Including a transmission function $T(\vec r)$:

$$\boxed{E_0(P)=\frac{k^i}{2\pi i}\int_S d\sigma\,\frac{e^{ik^0 R'}}{R'}\,e^{i\vec k^i\cdot\vec r}\,T(\vec r)\,E_i\cdot e^{i\phi(\vec r=0)}}$$

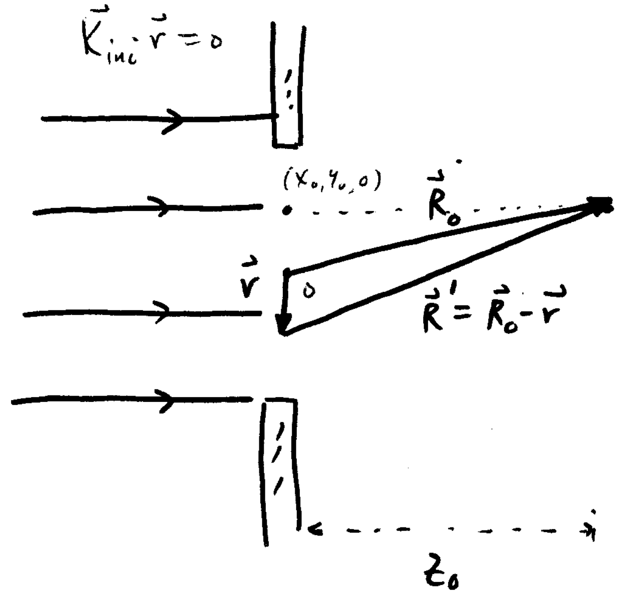
Kirchhoff - Fresnel Integral: (Fresnel, 1819, confirmed by Poisson)

(Principles of Optics, Max Born and Emil Wolf, p. 375-380, Eq. (17))

$$E(R_0) = E_{inc}\frac{-iK}{2\pi}\cdot\int_S d\sigma\, e^{i\vec{K}_{inc}\cdot\vec{r}}\,\frac{e^{iKR'}}{R'}\cdot\left(\frac{K_z^i + K_z^o}{2K}\right).$$

Let $\vec{R}_0 = (x_0, y_0, z_0)$,
$\vec{r} = (x, y, 0)$

We consider the situation when $z_0 \gg \lambda$ and the smallest dimension on S is larger compared to λ.



Geometric optics limit:

$$\phi(\vec{r}) \equiv KR' = \frac{2\pi}{\lambda}\sqrt{z_0^2 + (x-x_0)^2 + (y-y_0)^2}$$

is stationary around $x = x_0$, $y = y_0$, then increases rapidly so that the integral over (x, y) far away from (x_0, y_0) cancels each other. Because $z_0 \gg \lambda$, only those (x, y) that satisfies $|x-x_0| \ll z_0$, $|y-y_0| \ll z_0$ contribute to the integral:

$$\phi(\vec{r}) \simeq \frac{2\pi}{\lambda} z_0 + \frac{2\pi}{\lambda} \cdot \frac{(x-x_0)^2 + (y-y_0)^2}{2 z_0}$$

and the range of the integral is determined by $\sqrt{z_0 \lambda} \ll z_0$. Consequently,

$$E(R_0) \cong E_{inc} \frac{-ik}{2\pi} \frac{1}{z_0} e^{ikz_0} \cdot \iint_S dx\,dy\, e^{i \frac{k}{2z_0} \left[(x-x_0)^2 + (y-y_0)^2 \right]}$$

Since the dimension of S is large compared to λ, if it is also much larger than $\sqrt{\lambda z_0}$ (Geometric optics limit), then we can safely let the limit of the integral go to infinity:

$$E(R_0) = E_{inc} \frac{-ik}{2\pi} \frac{1}{z_0} e^{ikz_0} \int_{-\infty}^{+\infty} dx\, e^{i \frac{k}{2z_0} (x-x_0)^2} \int_{-\infty}^{\infty} dy\, e^{i \frac{k}{2z_0} (y-y_0)^2}$$

Changing variables to $x' = \sqrt{\frac{k}{2z_0}} (x - x_0)$, $y' = \sqrt{\frac{k}{2z_0}} (y - y_0)$, then

$$E(R_0) = E_{inc} \boxed{\frac{\cancel{k}}{\cancel{2\pi}}} \frac{-i}{\pi} e^{ikz_0} \int_{-\infty}^{+\infty} dx' e^{ix'^2} \int_{-\infty}^{+\infty} dy' e^{iy'^2}$$

$$= E_{inc} \frac{-i}{\pi} e^{ikz_0} \cdot \left(\int_{-\infty}^{+\infty} dx' e^{ix'^2} \right)^2$$

$$\int_{-\infty}^{+\infty} dx' e^{ix'^2} = 2\int_0^{+\infty} dx' e^{ix'^2} = \sqrt{\pi}\, e^{i\frac{\pi}{4}}$$

$$\therefore \left(\int_{-\infty}^{+\infty} dx' e^{ix'^2}\right)^2 = \pi e^{i\pi/2} = \pi i$$

$$\therefore \vec{E}(R_0) = \vec{E}_{inc} \cdot e^{ikz_0}$$, just as you expect.

If (x_0, y_0) is outside S by more than $\sqrt{\lambda z_0}$, then the strong cancellation in integration over (x, y) will result in a much reduced intensity at (x_0, y_0, z_0). Thus we see shadow.

Single long slit

(Geometric limit along x-direction)

$$\phi(\vec{r}) = \frac{2\pi}{\lambda}\sqrt{z_0^2 + (y-y_0)^2 + (x-x_0)^2}$$

$$\cong \frac{2\pi}{\lambda}\sqrt{z_0^2 + (y-y_0)^2} + \frac{2\pi}{\lambda}\cdot\frac{(x-x_0)^2}{2\sqrt{z_0^2+(y-y_0)^2}}$$

$$E\left(R=\sqrt{z_0^2+y_0^2}\right) = \frac{E_{inc}\,e^{-i\pi/2}}{\lambda}\int_{-d/2}^{d/2} dy\, \frac{e^{i\vec{k}_{inc}\cdot\vec{r}}\cdot e^{ik\sqrt{z_0^2+(y-y_0)^2}}}{\sqrt{z_0^2+(y-y_0)^2}}$$

$$\cdot\int_{-\infty}^{+\infty} dx\, e^{i\left(\frac{2\pi}{\lambda}\right)\cdot\frac{(x-x_0)^2}{\sqrt{z_0^2+(y-y_0)^2}}}$$

$$= \frac{E_{inc}\cdot e^{-i\pi/4}}{\sqrt{\lambda}}\int_{-d/2}^{d/2} dy\cdot\frac{e^{ik_{inc,y}\cdot y}\cdot e^{ik\sqrt{z_0^2+(y-y_0)^2}}}{\left(z_0^2+(y-y_0)^2\right)^{1/4}}$$

$$= \frac{E_{inc}\, e^{-i\pi/4}}{\sqrt{\lambda}}\int_{-d/2}^{d/2} dy\cdot\underbrace{\frac{e^{ik\sqrt{z_0^2+(y-y_0)^2}}}{\left(z_0^2+(y-y_0)^2\right)^{1/4}}}_{\text{cylindrical wavelet.}}\, e^{ik_{inc,y}\, y}$$